The Stability Analysis of Ecological Interaction of Multi-Species in Prey-Predator System

A. Umar[1] and **S. Musa**[1]
[1]*Department of Mathematics, Modibbo Adama University, Yola – Nigeria*

Correspondence: ahmedumar42@gmail.com.

Abstract

Honeybee keeping has been an agricultural practice that stabilizes natural ecosystems. Nevertheless, the honeybee population has been in constant decline worldwide due to enormous factors especially Varroa-mite infestation. The study developed a mathematical model to exhibits the interplay between three interacting species. In the interaction, Honeybee serves as a primary prey to Varroa-mite, and in turn Varroa-mite is a secondary prey to Pseudoscorpion. The introduction of pseudoscorpion as a super predator of Varroa-mite was a uniquely substantial mechanism to halt the negative impact of Varroa-mite infestation on the honeybee population. The existence of all the steady states alongside their local stability analysis prompted threshold conditions to guarantee the non-negative value of the species. The analytical result of all local stability analysis of the steady state confirmed that they were found to be Locally Asymptotically Stable (LAS). The study recommended that, using a super predator in an ecosystem with threshold conditions addressed and stabilized biological systems that exhibit cyclic behavior.

Keywords: *Steady State, Local Stability, Prey-Predator and Infestation.*

Introduction

Species interact in either ecological or biological processes, therefore, the dynamic relationships between these species have been obviously a complex phenomenon. The population dynamics of this vast species are a function of factors that are intrinsic to a population and the environmental conditions. Naturally, many factors such as food, space, and agents that reduce the species size inform of predation, and competition are indeed complex multidimensional problems. Thus, it becomes paramount to devise an avenue via intense studies and research to balance the ecosystem. The ecological interaction could be prey-predator, competition, mutualism, and commensalism.

The dynamic relationship between predators and their prey has long been and will continue to be one of the dominant themes in both ecology and mathematical ecology due to its universal existence and importance (Berryman, 1992). Thus, the need to develop and apply mathematical models through which species sustainability could be maintained, is on an invariable rise (Dubey & Upadhyay, 2004). To understand the ecological interaction of prey-predator, many authors had developed mathematical models such as Lotka and Volterra (Lotka, 1925 & Volterra, 1978). They proposed simple model of prey-predator interaction in which many more mathematical models had been introduced to understand more complex ecological interaction. In that light, MacArthur and Rosenzweig (1963) proposed a model of a community consisting of three species: super-predator, predator, and prey. The super-predator eats the predator which eats the prey, and both predators have Holling (or Monod) predation responses.

As ecosystem evolves with natural factors, researchers also get insights as to what components would be encapsulated into their models to mimic such natural vicissitudes. In line with that direction, vast studies were prompted such includes: Rayungsari, Kusumawinahyu and Marsudi (2004); Kar and Chaudhuri (2004); Pal and Samanta (2010); Misra, Sinha and Singh (2013); and among others. In addition, Rao, Rasappan, Murugesan and Srinivasan (2015) proposed a mathematical model of the dynamical complexities of a prey predator model with susceptible and infected (SI) prey with nonlinear functional response. With advance prey-predator model consisting three or more populations, Sayekti, Malik, and Aldila (2017) proposed an interaction between prey, secondary predator, and primary predator as a mathematical model of the one prey and two-predator system with constant harvesting in prey population. In the same fashion, Raymond, Hugo and Kung'aro (2019) proposed and studied the dynamics of two-prey one predator system of fishery model with Holling type II function response. The study also, examined the ecological dynamics of the Nile perch, cichlid, and tilapia fishes as prey-predator system of Lake Victoria fishery in Tanzania and the effect of harvesting.

Thus, this study intended to portray unique ecological interaction of honeybee, *Varroa-mite* and pseudoscorpion population with virus attacking honeybee as result of virus-carrying *Varroa-mite* infestation. Pseudoscorpion being a super predator on *Varroa-mite* and its impact on *Varroa-mite* brings a new ecological interaction model as well as a strategy to alleviate the negative impact of *Varroa-mite* on honeybee population.

Model Formulation

Assumptions

i. It was assumed that honeybee population grows in logistic manner
ii. It was assumed that *Varroa-mite* carrying virus infested (invaded) healthy honeybee only.
iii. It was assumed that *Varroa-mite* is a primary predator of Honeybee, while, Pseudoscorpion is a super predator of *Varroa-mite*.
iv. All population of these species were considered to die naturally.
v. It was also assumed that, viruses carried by *Varroa-mite* doesn't harm the super predator (Pseudoscorpion).
vi. It was assumed that, there is no competition among the super predators neither on the primary predators.

State variables and parameters of the model

The state variables and parameters of the model is presented in table 1 below:

Table 1: State Variables and Parameters

Symbol	Descriptions
$H(t)$	Population of honeybees at time t
$V(t)$	Population of *Varroa-mite* at time t
$P(t)$	Population of pseudoscorpion at time t
γ	Intrinsic growth rate of honeybee
k	Environmental carrying capacity of honeybee population
α_1	Infestation coefficient of *Varroa-mite* on honeybee
α_2	Capture coefficient of pseudoscorpion on *Varroa-mite*
b_1	Saturation rate of honeybee
b_2	Saturation rate of *Varroa-mite*
μ_1	Natural death rate of honeybee
μ_2	Natural death rate of *Varroa-mite*
μ_3	Natural death rate of pseudoscorpion
η_1	Conversion coefficient of honeybee to *Varroa-mite*
η_2	Conversion coefficient of *Varroa-mite* to pseudoscorpion
δ	**Virus induced death rate**

Schematic diagram of the model

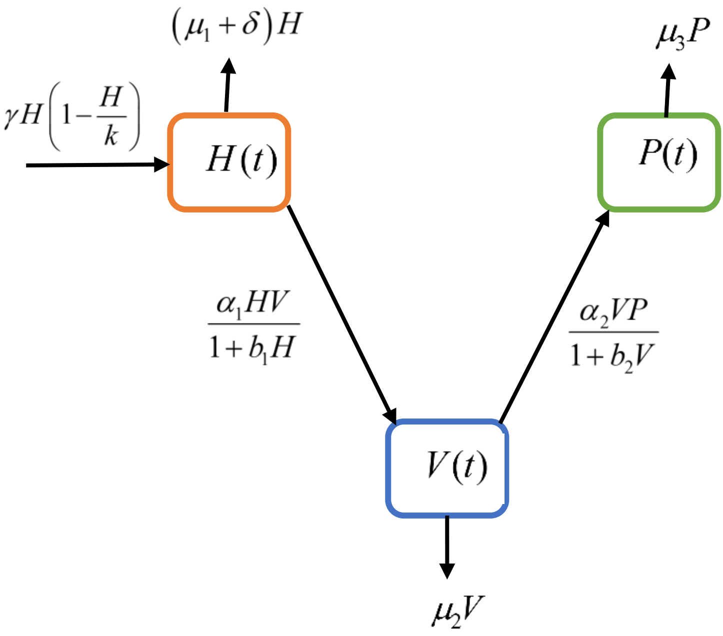

Figure 1: Schematic diagram of the multispecies interaction

Therefore, with the assumptions, state variables and parameters hitherto stated, gave the dynamic system of super predator, primary predator and a single prey. The prey is contracted with disease by factor of *Varroa-mite* being a vector of certain viruses. Thus, the dynamic system gives the following:

$$\left.\begin{aligned}
&\frac{dH}{dt}=\gamma H\left(1-\frac{H}{k}\right)-\frac{\alpha_1 HV}{1+b_1H}-\left(\mu_1+\delta\right)H\\
&\frac{dV}{dt}=\eta_1\frac{\alpha_1 HV}{1+b_1H}-\frac{\alpha_2 VP}{1+b_2V}-\mu_2 V\\
&\frac{dP}{dt}=\eta_2\frac{\alpha_2 VP}{1+b_2V}-\mu_3 P\\
&H(0)\geq 0,\ V(0)\geq 0,\ P(0)\geq 0
\end{aligned}\right\}\qquad(1)$$

Results and Discussions

Existence of the steady states of the model

In this section, the study established the existence of the steady states of the model by setting the system of equation (1) and the dynamical system was found to have eight (8) steady states (one trivial and seven nontrivial) viz: $E_1\left(0,0,0\right)$, $E_2\left(H^*,0,0\right)$, $E_3\left(0,V^*,0\right)$, $E_4\left(0,0,P^*\right)$, $E_5\left(H^*,V^*,0\right)$, $E_6\left(H^*,0,P^*\right)$, $E_7\left(0,V^*,P^*\right)$ and $E_8\left(H^*,V^*,P^*\right)$. The analysis of the existence of each of the steady state is gives as follows:

Extinction of the three populations $E_1\left(0,0,0\right)=\left(0,0,0\right)$

This trivial steady state implies the absence of the population of honeybee, *Varroa-mite* and pseudoscorpion. Hence, there is absolute nonexistence of ecological interaction of these populations.

Existence of honeybee only $E_2\left(H^*,0,0\right)$ **with** $H^*>0$.

Let $V=0$ and $P=0$ in equation (1), then equation (1) becomes $\gamma H\left(1-\frac{H}{k}\right)-\left(\mu_1+\delta\right)H=0$, from which it gives $H^*=\frac{k\left(\gamma-\mu_1-\delta\right)}{\gamma}$.

Hence $E_2\left(H^*,0,0\right)=\left(\frac{k\left(\gamma-\mu_1-\delta\right)}{\gamma},0,0\right)$.

Therefore, the steady state E_2 exists if $\gamma>\left(\mu_1+\delta\right)$ (2)

Condition (2) implies that the intrinsic growth rate of honeybee must exceed the sum of natural and virus induced death rate for the steady state E_2 to exist.

Existence of *Varroa-mite* only $E_3\left(0,V^*,0\right)$ **with** $V^*>0$

Let $H=0$, and $P=0$. Then equation (1) becomes $\frac{dV}{dt}=-\mu_2 V$, from which we have $V(t)=V_0e^{-\mu_2 t}$ as the solution. As such *Varroa-mite* population in the absence of honeybee as $t\to\infty$, $V(t)\to 0$. Hence $E_3\left(0,V^*,0\right)$ exist. But it has a zero-limiting value; which interpret that, in the absence of honeybee, *Varroa-mite* population go to extinction over time.

Existence of pseudoscorpion only $E_4\left(0,0,P^*\right)$ **with** $P^*>0$

Let $H=0$, and $V=0$. Then equation (1) becomes $\frac{dP}{dt}=-\mu_3 P$, from which we have $P(t)=P_0e^{-\mu_3 t}$ as the solution.

Therefore, pseudoscorpion population in the absence of *Varroa-mite* as $t \to \infty$, $P(t) \to 0$. Therefore, pseudoscorpion cannot continue to exist over time in the absence of *varroa-mite*. Hence, $E_4\left(0,0,P^*\right)$ exist.

Existence of interaction between honeybee and *Varroa-mite* $E_5\left(H^*,V^*,0\right)$ with $H^*>0$ *and* $V^*>0$.

Let $P=0$. Then equation (1) becomes

$$\left.\begin{aligned} &\gamma H\left(1-\frac{H}{k}\right)-\frac{\alpha_1 HV}{1+b_1H}-\left(\mu_1+\delta\right)H=0 \\ &\eta_1\frac{\alpha_1 HV}{1+b_1H}-\mu_2 V=0 \end{aligned}\right\} \quad (3)$$

Solving equation (3) simultaneously, it gives:

$$H^*=\frac{\mu_2}{\alpha_1\eta_1-b_1\mu_2} \text{ and}$$

$$V^*=\frac{\eta_1(k(\delta+\mu_1)(b_1\mu_2-\alpha_1\eta_1)-\gamma(b_1k\mu_2-\alpha_1k\eta_1+\mu_2)}{k(b_1\mu_2-\alpha_1\eta_1)^2} \quad (4)$$

Thus, E_5 exist if the following conditions hold:

$\dfrac{\alpha_1\eta_1}{b_1}>\mu_2$ in H^*, while $b_1\mu_2>\alpha_1\eta_1$, and

$$k(\delta+\mu_1)(b_1\mu_2-\alpha_1\eta_1)>\gamma(b_1k\mu_2-\alpha_1k\eta_1+\mu_2) \text{ in } V^* \quad (5)$$

If these conditional inequalities in (5) hold. Then, steady state E_5 exist. These conditions imply that, the coexistence between honeybee and *Varroa-mite* is confirmed and steady state E_5 exist.

Existence of interaction between honeybee and pseudoscorpion $E_6\left(H^*, 0, P^*\right)$ with $H^* > 0$ *and* $P^* > 0$.

Let $V = 0$. Then, equation (1) becomes

$$\left.\begin{aligned} &\gamma H\left(1-\frac{H}{k}\right)-\left(\mu_1+\delta\right)H=0 \\ &-\mu_2 V=0 \end{aligned}\right\} \quad (6)$$

Solving equation (6) simultaneously gives:

$$H^* = \frac{k(\gamma-\delta-\mu)}{\gamma} \text{ and } P^* = 0 \qquad (7)$$

Thus, the steady state E_6 exist if $\gamma > \left(\delta+\mu\right)$ condition hold in equation (7), which implies that intrinsic growth rate of honeybee exceeds the sum of natural and virus induced death rates.

Existence of interaction between *Varroa-mite* and pseudoscorpion $E_7\left(0, V^*, P^*\right)$ with $V^* > 0$ *and* $P^* > 0$.

Let $H = 0$. Then, equation (1) becomes

$$\left.\begin{aligned} &-\frac{\alpha_2 VP}{1+b_2 V} - \mu_2 V = 0 \\ &\eta_2 \frac{\alpha_2 VP}{1+b_2 V} - \mu_2 V = 0 \end{aligned}\right\} \quad (8)$$

Solving equation (8) simultaneously gives:

$$V^* = \frac{\mu_3}{\alpha_2\eta_2 - \mu_3 b_2} \text{ and } P^* = \frac{\eta_2\mu_2}{\mu_3 b_2 - \alpha_2\eta_2} \quad (9)$$

Therefore, the steady state E_7

exist if $\frac{\alpha_2\eta_2}{b_2} > \mu_3$ in V^* and $b_2\mu_3 > \alpha_2\eta_2$ in P^* holds in equation (9).

Existence of interior (co-existence) steady state $E_8\left(H^*, V^*, P^*\right)$

The three distinct population of honeybee, *Varroa-mite* and pseudoscorpion interact freely in food chain system for survival and reproduction. As such, the interior steady state E_8 exist by solving the system of equation (1) and it gives:

$$H^* = \frac{1}{2}\left(-D + C\frac{\sqrt{E}}{b_1\gamma(\alpha_2\eta_2 - \mu_3 b_2)}\right) \quad (10)$$

Where:

$$E=(\alpha_2\eta_2-\mu_3b_2)\left((\alpha_2\eta_2-\mu_3b_2)(-b\gamma k+bk(\delta+\mu_1)+\gamma)^2-4b\gamma k(\mu_3(\alpha_1+b_2(\gamma-\delta-\mu_1))+\alpha_2\eta(-\gamma+\delta+\mu_1))\right)$$

$D=\frac{1}{b_1}$, and $C=\frac{k(\gamma-\delta-\mu_1)}{\gamma}$

$$V^*=\frac{\mu_3}{\alpha_2\eta_2-\mu_3b_2} \tag{11}$$

$$P^*=\frac{\eta_2\left(A+\eta_1\sqrt{B}\right)}{2\mu_3b_1^2k(\mu_3b_2-\alpha_2\eta_2)} \tag{12}$$

$$B=(\alpha_2\eta_2-\mu_3b_2)\left((\alpha_2\eta_2-\mu_3b_2)(-b\gamma k+bk(\delta+\mu_1)+\gamma)^2-4b\gamma k(\mu_3(\alpha_1+b_2(\gamma-\delta-\mu_1))+\alpha_2\eta_2(-\gamma+\delta+\mu_1))\right)$$

$$A=2\mu_3b_1^2k\mu_2+\eta_1(bk(-2\mu_3\alpha_1+\alpha_2\eta_2(\gamma-\delta-\mu_1)+\mu_3b_2(-\gamma+\delta+\mu_1))+\gamma\alpha_2\eta_2-\mu_3\gamma b_2)$$

Therefore, the steady state E_8 exist if the following conditions hold:

$$\frac{\alpha_2\eta_2}{b_2}>\mu_3 \text{ in } H^* \text{ and } V^*\text{, while } b_2\mu_3>\alpha_2\eta_2 \text{ in } P^* \tag{13}$$

Hence, the steady state E_8 exist if the inequalities in (13) hold respectively on equation (11) and (12). The biological interpretation implies that, the ratio of product of capture and conversion coefficient of pseudoscorpion by its saturation must exceed its death rate.

Local stability of the steady state $E_1\left(0,0,0\right)$

The Jacobian matrix evaluated at the steady state E_1 is given as

$$J\left(E_1\right)=\begin{pmatrix}\gamma-\mu_1-\delta & 0 & 0\\ 0 & -\mu_2 & 0\\ 0 & 0 & -\mu_3\end{pmatrix} \tag{14}$$

The eigenvalues of $J\left(E_1\right)$ are: $\gamma-\mu_1-\delta$, $-\mu_2$ and $-\mu_2$. The computed eigenvalues are negative if the first eigenvalue satisfies $\gamma<\left(\mu_1+\delta\right)$. Hence, the steady state E_1 is locally asymptotically stable.

Local stability of the steady state $E_2\left(H^*,0,0\right)$

The Jacobian matrix evaluated at the steady state E_2 is given as

$$J\left(E_2\right)=\begin{pmatrix} \gamma-\frac{2H^*\gamma}{k}-\mu_1-\delta & -\frac{\alpha_1 H^*}{1+b_1H^*} & 0 \\ 0 & \eta_1\frac{\alpha_1 H^*}{1+b_1H^*}-\mu_2 & 0 \\ 0 & 0 & -\mu_3 \end{pmatrix} \qquad (15)$$

The eigenvalues of $J\left(E_2\right)$ are: $-\gamma-3\mu_1-3\delta$, $\eta_1\alpha_1H^*>\mu_2\left(1+b_1H^*\right)$ and $-\mu_2$. The computed eigenvalues are all negative if the second eigenvalue satisfies the inequality condition and so, the steady state E_2 is locally asymptotically stable.

Local stability of the steady state $E_3\left(0,V^*,0\right)$

The Jacobian matrix evaluated at the steady state E_3, it gives:

$$J\left(E_3\right)=\begin{pmatrix} \gamma-\mu_1-\delta & 0 & 0 \\ \eta_1\alpha_1V^* & -\mu_2 & 0 \\ 0 & 0 & \eta_2\frac{\alpha_2V^*}{1+b_2V^*}-\mu_3 \end{pmatrix} \qquad (16)$$

The eigenvalues of the matrix $J\left(E_3\right)$ are: $\gamma-\mu_1-\delta$, $-\mu_2$ and $\frac{\alpha_2 V^*}{1+b_2 V^*}-\mu_3$. The first eigenvalue is negative, if it satisfies this inequality $\gamma<\left(\mu_1+\delta\right)$. The second eigenvalue is negative. The third eigenvalue is $\eta_2 \frac{\alpha_2 V^*}{1+b_2 V^*}<\mu_3$,

such that

$$V^*=\frac{\mu_3}{\left(\frac{\eta_2\alpha_2}{b_2}\right)-\mu_3} \tag{17}$$

provided that $\left(\frac{\eta_2\alpha_2}{b_2}\right)<\mu_3$ in equation (17) holds. Hence, the steady state E_3 is stable

Local stability of the steady state $E_4\left(0,0,P^*\right)$

The Jacobian matrix evaluated at the steady state E_4, we have:

$$J\left(E_4\right)=\begin{pmatrix} \gamma-\mu_1-\delta & 0 & 0 \\ 0 & -\mu_2 & 0 \\ 0 & \eta_2\alpha_2 P^* & -\mu_3 \end{pmatrix} \tag{18}$$

The eigenvalues of the matrix $J\left(E_4\right)$ are: $\gamma-\mu_1-\delta$, $-\mu_2$ and $-\mu_3$.
The eigenvalues above are negative if for

$$\gamma<\left(\mu_1+\delta\right) \tag{19}$$

Hence, the steady state E_4 is locally asymptotically stable if inequalities in (19) hold.

Local stability of the steady state $E_5\left(H^*, V^*, 0\right)$

The Jacobian matrix evaluated at the steady state E_5, it gives:

$$J\left(E_5\right)=\begin{pmatrix} \gamma-\frac{2H^*\gamma}{k}+\frac{\alpha_1 H^* V^* b_1}{\left(1+b_1H^*\right)}-\frac{\alpha_1 V^*}{\left(1+b_1H^*\right)}-\mu_1-\delta & \frac{-\alpha_1 H^*}{\left(1+b_1H^*\right)} & 0 \\ \eta_1\frac{\alpha_1 H^* V^*}{\left(1+b_1H^*\right)^2}+\frac{\eta_1\alpha_1 V^*}{\left(1+b_1H^*\right)} & \eta_1\frac{\alpha_1 H^*}{\left(1+b_1H^*\right)}-\mu_2 & \frac{\alpha_2 V^*}{\left(1+b_2V^*\right)} \\ 0 & 0 & \eta_2\frac{\alpha_2 V^*}{\left(1+b_2V^*\right)}-\mu_3 \end{pmatrix} \quad (20)$$

The eigenvalues of the matrix $J\left(E_5\right)$ are:

$\gamma-\frac{2H^*\gamma}{k}+\frac{\alpha_1 H^* V^*}{1+b_1H^*}-\frac{\alpha_1 V^*}{1+b_1H^*}-\mu_1-\delta$, $\eta_1\frac{\alpha_1 H^*}{1+b_1H^*}-\mu_2$ and $\eta_2\frac{\alpha_2 V^*}{1+b_2V^*}-\mu_3$.

The eigenvalues are negative if the following conditions are satisfied:

$$\left(\gamma+\frac{\alpha_1 H^* V^*}{1+b_1H^*}\right)<\left(\frac{2H^*\gamma}{k}+\frac{\alpha_1 V^*}{1+b_1H^*}+\mu_1+\delta\right) \quad (21)$$

$$\eta_1\alpha_1 H^*<\left(1+b_1H^*\right)\mu_2 \quad (22)$$

$$\eta_2\alpha_2 V^*<\left(1+b_2V^*\right)\mu_3 \quad (23)$$

If all the conditions in (21), (22) and (23) hold respectively, the steady state E_5 is locally asymptotically stable.

Local stability of the steady state $E_6\left(H^*,0,P^*\right)$

The Jacobian matrix evaluated at the steady state E_6, it gives:

$$J\left(E_6\right)=\begin{pmatrix} \gamma-\frac{2\gamma H^*}{k}-\mu_1-\delta & -\frac{\alpha_1 H^*}{1+b_1 H^*} & 0 \\ 0 & \eta_1\frac{\alpha_1 H^*}{1+b_1 H^*}-\mu_2 & 0 \\ 0 & 0 & -\mu_3 \end{pmatrix} \tag{24}$$

The eigenvalues of the matrix

$J\left(E_6\right)$ are: $\gamma-\frac{2\gamma H^*}{k}-\mu_1-\delta$, $\eta_1\frac{\alpha_1 H^*}{1+b_1 H^*}-\mu_2$ and $-\mu_3$.

The eigenvalues are negative if the following conditions are satisfied:

$$\gamma<\left(\frac{2\gamma H^*}{k}+\mu_1+\delta\right) \tag{25}$$

$$\eta_1\alpha_1 H^*>\left(1+b_1 H^*\right)\mu_2 \tag{26}$$

If all the conditions in (25) and (26) hold respectively, the steady state E_6 is locally asymptotically stable.

Local stability of the steady state $E_7\left(0,V^*,P^*\right)$

The Jacobian matrix evaluated at the steady state E_7, it gives:

$$J\left(E_7\right)=\begin{pmatrix} \gamma-\mu_1-\delta-\alpha_1V^* & 0 & 0 \\ \eta_1\alpha_1V^* & -\mu_2 & \dfrac{\alpha_2V^*}{\left(1+b_2V^*\right)} \\ 0 & -\dfrac{\eta_2\alpha_2V^*P^*b_2}{\left(1+b_2V^*\right)^2}+\dfrac{\eta_2\alpha_2P^*}{\left(1+b_2V^*\right)} & \dfrac{\eta_2\alpha_2V^*}{\left(1+b_2V^*\right)}-\mu_3 \end{pmatrix} \tag{27}$$

The eigenvalues of the matrix $J\left(E_7\right)$ are: $\gamma-\mu_1-\delta+\alpha_1V^*$, $-\mu_2$ and $\dfrac{\eta_2\alpha_2V^*}{\left(1+b_2V^*\right)}-\mu_3$. The eigenvalues are negative if the following conditions are satisfied:

$$\left(\gamma+\alpha_1V^*\right)<\left(\mu_1+\delta\right) \tag{28}$$

$$-\mu_3<0 \tag{29}$$

$$\eta_2\alpha_2V^*<\left(1+b_2V^*\right)\mu_3 \tag{30}$$

If all the conditions in (28), (29) and (30) hold respectively, the steady state E_7 is locally asymptotically stable.

Local stability of the steady state $E_8\left(H^*,V^*,P^*\right)$

The Jacobian matrix evaluated at the steady state E_8, it gives:

$$J\left(E_8\right)=\begin{pmatrix} \gamma-\frac{2H^*\gamma}{k}+\frac{\alpha_1H^*V^*b_1}{\left(1+b_1H^*\right)}-\frac{\alpha_1V^*}{\left(1+b_1H^*\right)}-\mu_1-\delta & -\frac{\alpha_1H^*}{\left(1+b_1H^*\right)} & 0 \\ -\frac{\eta_1\alpha_1H^*V^*b_1}{\left(1+b_1H^*\right)^2}+\frac{\eta_1\alpha_1V^*}{\left(1+b_1H^*\right)} & \eta_1\frac{\alpha_1H^*}{\left(1+b_1H^*\right)}-\mu_2 & \frac{\alpha_2V^*}{\left(1+b_2V^*\right)} \\ 0 & -\frac{\eta_2\alpha_2V^*P^*b_2}{\left(1+b_2V^*\right)^2}+\frac{\eta_2\alpha_2P^*}{\left(1+b_2V^*\right)} & \frac{\eta_2\alpha_2V^*}{\left(1+b_2V^*\right)}-\mu_3 \end{pmatrix} \qquad (31)$$

Rewriting matrix (31) and getting the characteristics equation, it gives:

$$J\left(E_8\right)=\begin{vmatrix} m_1-\lambda & m_2 & 0 \\ m_3 & m_4-\lambda & m_5 \\ 0 & m_6 & m_7-\lambda \end{vmatrix}=0 \qquad (32)$$

Where:

$$m_1=\gamma-\frac{2H^*\gamma}{k}+\frac{\alpha_1H^*V^*b_1}{\left(1+b_1H^*\right)}-\frac{\alpha_1V^*}{\left(1+b_1H^*\right)}-\mu_1-\delta, \qquad m_2=\frac{\alpha_1H^*}{\left(1+b_1H^*\right)},$$

$m_4=\eta_1\frac{\alpha_1H^*}{\left(1+b_1H^*\right)}-\mu_2$, $m_5=\frac{\alpha_2V^*}{\left(1+b_2V^*\right)}$, $m_6=-\frac{\eta_2\alpha_2V^*P^*b_2}{\left(1+b_2V^*\right)^2}+\frac{\eta_2\alpha_2P^*}{\left(1+b_2V^*\right)}$ and

$$m_7=\frac{\eta_2\alpha_2V^*}{\left(1+b_2V^*\right)}-\mu_3$$

The local stability of (32) is determined by the polynomial characteristics given below:

$$\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 = 0 \tag{33}$$

Where:

$a_1 = \left(-m_1 - m_4 - m_7\right)$, $a_2 = \left[m_7\left(m_1 + m_4\right) + m_1m_4 - m_2m_3 - m_5m_6\right]$ and

$a_3 = \left[m_5m_6\left(m_1 + m_4\right) - m_7\left(m_1m_4 - m_2m_3\right) - m_4m_5m_6\right]$

By the Routh-Hurwitz criteria, it follows that all eigenvalues of equation (33) have negative real parts if and only if

$a_1 > 0,\ a_3 > 0$ and $a_1a_2 > a_3$ (34)

Satisfying condition (34) indicated that, the steady state E_8 is locally asymptotically stable.

Lastly, the model had shown that, the existence of only honeybee overtime, the intrinsic growth rate of honeybee must exceed the sum of natural and virus induced death rate which was the same interpretation in Ismael and Majeed (2019), and Ganguli and Kar (2017). Both *Varroa-mite* and pseudoscorpion existing in isolation, has a zero-limiting value which translated that, they are obligated predator to their respective prey as revealed in (Hadžiabdić, Mehuljić & Bektešević, 2017). Moreso, the coexistence of prey (Honeybee) and predator (*Varroa-mite*) or prey (*Varroa-mite*) and predator (Pseudoscorpion) is possible, but the interactions revealed a cyclic behaviour which could cause crash, such phenomena can be halted if conditions were set to stabilized the population and prevented the cyclic behaviour as confirmed by

(Kar & Batabyal, 2010). Furthermore, the study had shown that, the predator's mortality rate and food conversion coefficients had significantly influenced the stability behaviour of the multispecies interaction as compared to Dubey and Upadhyay (2004). The local stability of the eight steady states with their corresponding conditions were found to be locally and asymptotically stable.

Conclusion

The mathematical model for honeybee, *Varroa-mite* and pseudoscorpion in prey-predator interaction was proposed and analysed. The existence of all the steady states alongside their local stability analysis prompted threshold conditions to guarantee the non-negative value of the species. Apparently, the analytical results of the model portrayed that, coexistence of honeybee, *Varroa-mite* and pseudoscorpion populations is feasible; given the pressure of virus infection and *Varroa-mite* infestation on honeybee population. Perceptively in this study, such ecological interaction of the species is quite different as compared with other prey-predator models that exhibit oscillatory pattern in their qualitative behaviour. Thus, in this study, the prey-predator interplay is substantially stabilized consequence to pseudoscorpion existence in the system.

References

Berryman, A. A. (1992). The Origins and Evolutions of Predator–Prey Theory. *Ecology* 73(5):1530–535.

Dubey, B., & Upadhyay, R. K. (2004). Persistence and extinction of one-prey and two-predators system. *Nonlinear Analysis: Modelling and Control*, *9*(4), 307-329.

Ganguli, C., Kar, T. K., & Mondal, P. K. (2017). Optimal harvesting of a prey–predator model with variable carrying capacity. *International Journal of Biomathematics*, *10*(05), 1750069.

Hadžiabdić, V., Mehuljić, M., & Bektešević, J. (2017). Lotka-volterra model with two predators and their prey. *Tem Journal*, *6*(1), 2217-8309. http://doi:10.18421/TEM61-19, February 2017

Ismaeel, M. H., & Majeed, A. A. (2019, November). A proposed bio-economic model for the prey-predator fishery model with harvesting: toward modeling dynamics in applied mathematics. In *Journal of Physics: Conference Series* (Vol. 1362, No. 1, p. 012142). IOP Publishing.

Kar, T. K., & Chaudhuri, K. S. (2004). Harvesting in a Two-Prey One Predator Fishery: A Bioeconomic Model. *e ANZIAM Journal*, *45*(3):443–456.

Kar, T. K., & Batabyal, A. (2010). Persistence and stability of a two prey one predator system. *International Journal of Engineering, Science and Technology*, 2(2):174-190

Lotka, A. J. (1925). *Elements of Physical Biology*. Williams and Walkins Company, Baltimore, Maryland.

MacArthur, R. H., & Rosenzweig, M. L. (1963). Graphical Representation and Stability Conditions of Predator-Prey Interactions. *The American Naturalist*. *97* (1963): 209.

Misra, O. P., Sinha, P., & Singh, C. (2013). Stability and Bifurcation Analysis of a Prey–Predator Model with Age Based Predation. *Applied Mathematical Modelling*. *37*:6519–6529. Journal homepage: www.elsevier.com/locate/apm

Pal, A. K., & Samanta, G. P. (2010). Stability Analysis of an Eco-epidemiological Model Incorporating a Prey Refuge. *Nonlinear Analysis: Modelling and Control*, *15*(4):473–491

Rao, V. L. G. M., Rasappan, S., Murugesan, R. & Srinivasan, V. (2015). A Prey Predator Model with Vulnerable Infected Prey Consisting of Non-Linear Feedback. *Applied Mathematical Sciences*. *9*(42):2091-2102. http://.doi.org/10.12988/ams.2015.43184

Raymond, C., Hugo, A., & Kung'aro, M. (2019). Modeling Dynamics of Prey-Predator Fishery Model with Harvesting:

A Bioeconomic Model. *Journal of Applied Mathematics* Volume 2019, Article ID 2601648, 13 pages https://doi.org/10.1155/2019/2601648

Rayungsari, M. Kusumawinahyu, M. W., & Marsudi (2004). Dynamical Analysis of Predator-Prey Model with Ratio-Dependent Functional Response and Predator Harvesting. *Applied Mathematical Sciences*. *8*(29), 1401–1410 http://dx.doi.org/10.12988/ams.2014.4111

Sayekti, I. M., Malik, M., & Aldila, D. (2017). One-Prey Two-Predator Model with Prey Harvesting in a Food Chain Interaction. In AIP Conference Proceedings 1862, 030124 (2017); https://doi.org/10.1063/1.4991228 Published Online: 10 July 2017

Volterra, V. (1978) in *Lecture Notes in Biomathematics: Variation and Fluctuations in the Number of Coexisting Animal Species* (Springer, Berlin, 1978). 22:65–236.